ELSEVIER
Discrete Applied Mathematics 118 (2002) 239–248
DISCRETE APPLIED MATHEMATICS

On the computational complexity of strong edge coloring

Mohammad Mahdian

Department of Mathematics, MIT, 77 Massachusetts Avenue, Rm 2-342, Cambridge, MA 02139-4307, USA

Received 17 August 2000; received in revised form 8 March 2001; accepted 19 March 2001

Abstract

In the strong edge coloring problem, the objective is to color the edges of the given graph with the minimum number of colors so that every color class is an induced matching. In this paper, we will prove that this problem is **NP**-complete even in a very restricted setting. Also, a closely related problem, namely the maximum antimatching problem, is studied, and some **NP**-completeness results and a polynomial time algorithm for a subproblem are derived.

Keywords: Strong edge coloring; Antimatching; Induced matching; **NP**-completeness

1. Introduction

A *strong edge coloring* of a graph G is an assignment of colors to the edges of G such that every two edges *of distance at most two* receive different colors. Two edges are of distance at most two iff either they share an endpoint, or an endpoint of one is joined to an endpoint of the other. In other words, a strong edge coloring of G is a partition of $E(G)$ into a collection of induced matchings.

In 1985, Erdös and Nešetřil (see [4,5]) conjectured that every graph of maximum degree Δ has a strong edge coloring with at most $\frac{5}{4}\Delta^2$ colors. Also, it is conjectured by Faudree et al. [4] that every bipartite graph of maximum degree Δ has a strong edge coloring with Δ^2 colors. These conjectures are still wide open. For a summary of results concerning strong edge coloring refer to [13].

The computational complexity of finding a strong edge coloring with the minimum number of colors is studied for chordal graphs in [2], and a polynomial time algorithm for this problem, and another related problem, namely the problem of finding the largest induced matching in the given graph, is presented. It is also proved that the induced matching problem is **NP**-complete even for bipartite graphs. In fact, a slightly modified

E-mail address: mahdian@mit.edu (M. Mahdian).

0166-218X/02/$ - see front matter
PII: S0166-218X(01)00237-2

version of the argument in [2] implies that for every g, the induced matching problem is **NP**-complete for bipartite graphs of girth at least g. In this paper, we will prove that the strong edge coloring problem is **NP**-complete even if the graph is restricted to be bipartite and of girth at least g.

The strong edge coloring problem is, in fact, a special case of the problem of coloring the square of the given graph; it is coloring the square of the line-graph of the graph (see [2]). The problem of coloring the square of a graph is called *distance*-2 *coloring*, and has many applications, for example in approximating the Hessian matrices of certain nonlinear functions using a minimum number of gradient evaluations (see [14]), or in frequency assignment (broadcast scheduling) in packet radio networks (see [15]). This problem is well studied and is proved to be **NP**-complete even for planar graphs of bounded degree. See [13] for a summary of results.

Another related problem is the antimatching problem. An *antimatching* in a graph G is a set A of edges of G such that any two edges in A are of distance at most two. (In [2], antimatchings were called neighborly sets.) Therefore, the size of the largest antimatching in G is obviously a lower bound for the number of colors necessary for a strong edge coloring of G. We will prove that the antimatching problem is **NP**-complete for general graphs and for bipartite multigraphs, and polynomially solvable for graphs which do not contain any cycle of length 4, denoted by C_4.

The rest of this paper is organized as follows. In Sections 2 and 4, we will prove the **NP**-completeness of the strong edge coloring problem and the antimatching problem. Some comments about the strong edge coloring of special classes of graphs are mentioned in Section 3. In Section 5, a polynomial-time algorithm for the antimatching problem for C_4-free graphs is presented. Section 6 contains some open problems.

2. The strong edge coloring problem

In this section, we prove that the strong edge coloring problem is **NP**-complete. The problem is defined as follows:

STRONG EDGE COLORING

Instance: A graph G and a positive integer K.

Question: Does G have a strong edge coloring with K colors?

For $K \leqslant 3$, the problem is trivial (It is not difficult to see that the only strong 3-edge-colorable graphs are disjoint unions of cycles (of lengths divisible by 3), paths, and $K_{1,3}$'s). The following theorem implies that the problem is **NP**-complete for any other value of K.

Theorem 1. *For every fixed g, $K \geqslant 4$,* STRONG EDGE COLORING *is* **NP**-*complete for bipartite graphs with girth at least g.*

Proof. It is clear that the problem is in **NP**. We prove that it is **NP**-complete by showing a reduction from GRAPH K-COLORABILITY. It is well known [6] that GRAPH

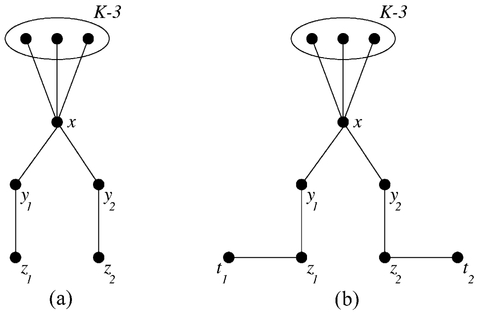

Fig. 1. (a) The graph T, (b) The graph T'.

K-COLORABILITY is **NP**-complete for every fixed $K \geqslant 3$. For every $K \geqslant 4$ and every graph H, we construct another graph G such that H is K-colorable if and only if G has a strong edge coloring with K colors.

Before starting to construct G, consider the graph T shown in Fig. 1(a). This graph consists of a vertex x adjacent to a set of $K-1$ other vertices, including two vertices y_1 and y_2, which are adjacent to two other vertices z_1 and z_2, respectively. The important property of this graph, which will be used throughout the proof, is that in every strong edge coloring of T with K colors, the colors of the two edges y_1z_1 and y_2z_2 are the same.

Now, for every $K \geqslant 4$, $d \geqslant 1$, we construct a graph $G_{K,d}$. $G_{K,d}$ contains a path $v_0v_1 \dots v_{3(d-1)}$. For every $i = 0, \dots, d-1$, the vertex v_{3i} is connected to another vertex u_i, and u_i is connected to another vertex w_i. Also, for every $i = 0, \dots, d-1$, the vertex v_{3i} is connected to a set A_i of new vertices (A_i's are pairwise disjoint). A_0 and A_{d-1} each contain $K-3$ vertices, and A_i (for $i = 1, \dots, d-2$) contains $K-4$ vertices. There is no other vertex or edge in $G_{K,d}$. For example, Fig. 2 shows the graph $G_{6,4}$. The vertices $w_0, w_1, \dots, w_{d-1}$ are called the *heads* of $G_{K,d}$.

Using the property of the graph T, it is easy to see that in every strong edge coloring of $G_{K,d}$ using K colors, the colors of the edges which are indicated by thick lines in Fig. 2 are the same.

Now, we are ready to construct the graph G from the graph H. Corresponding to each vertex v of degree d in H, we put a copy C_v of $G_{K,d}$ in G. Each head of C_v corresponds to one of the edges incident to v. If two vertices u and v in H are joined by an edge e, we combine the heads corresponding to e in C_u and C_v into a single vertex. Let G be the resulting graph. We claim that G has a strong edge coloring with K colors if and only if H is K-colorable.

Assume that G has a strong edge coloring with K colors. By the property of $G_{K,d}$, we know that for every vertex $v \in V(H)$ the colors of the thick edges of C_v are the same. Color the vertex v in H with the color of the thick edges of C_v. It is obvious that this coloring is a proper vertex coloring of H using K colors.

Conversely, assume that H has a vertex coloring using K colors. We construct a strong edge coloring of G using K colors. For every v, we color the thick edges of

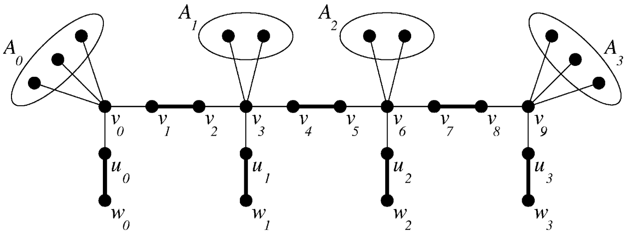

Fig. 2. The graph $G_{6,4}$.

C_v with the color of v in H. Then, we color the remaining edges of C_v using the following lemma.

Lemma 1. *Let T' be the graph obtained from T (Fig. 1(a)) by adding two new vertices t_1 and t_2 and connecting them to z_1 and z_2, respectively (see Fig. 1(b)). If the edges y_1z_1, y_2z_2, z_1t_1, and z_2t_2 are colored in such a way that the colors of y_1z_1 and y_2z_2 are the same and every two edges of distance at most two are colored with different colors, then it is possible to complete this partial coloring to a strong edge coloring of T'.*

Proof. It is sufficient to consider two cases: In the first case, z_1t_1 and z_2t_2 are colored with the same color, and in the second case, they are colored with different colors. In each case, it is easy to complete the coloring. □

Using the above lemma, we can color every edge in G, and obtain a strong edge coloring of G with K colors.

Therefore, we have proved that for every fixed $K \geqslant 4$, GRAPH K-COLORABILITY reduces to STRONG EDGE COLORING. The only thing that remains is to modify the reduction in such a way that the resulting graph G becomes bipartite and of girth at least g.

In order to do this, it is sufficient to consider the graph $G'_{K,d} = G_{K,2dg}$, and let the *heads* of $G'_{K,d}$ be the vertices $w_0, w_{2g}, w_{4g}, \ldots, w_{2g(d-1)}$ (i.e., w_{2gi} for $i = 0, \ldots, d-1$), where $w_0, w_1, \ldots, w_{2gd-1}$ are the heads of $G_{K,2dg}$. It is easy to see that if instead of $G_{K,d}$, we use $G'_{K,d}$ in the above reduction, the resulting graph G will be bipartite and has girth more than g. Furthermore, G has a strong edge coloring using K colors if and only if H is K-colorable. □

Notice that in particular the above theorem implies that STRONG EDGE COLORING is **NP**-complete for C_4-free graphs.

3. Strong edge coloring problem on special classes of graphs

The induced matching problem is investigated for many special classes of graphs, and for many classes, polynomial time algorithms are found. Most of these algorithms are based on the fact that the maximum induced matching in a graph G corresponds to the maximum independent set in the square of the line graph of G (see [2]). For many classes of graphs (e.g., chordal graphs, circular arc graphs, co-comparability graphs, and trapezoid graphs), it is proved that if a graph G is in the class $\mathscr{C}$, then the square of the line graph of G is also in $\mathscr{C}$ (see [2,9]). Therefore, if we can solve the maximum independent set problem in class $\mathscr{C}$, we can also solve the induced matching problem in this class.

The same idea can be applied to the strong edge coloring problem, because a strong edge coloring of a graph G is a proper vertex coloring of the square of the line graph of G. Using this fact, we can obtain polynomial time algorithms for strong edge coloring of chordal graphs (see [2]) and co-comparability graphs [9,7]. The latter class contains interval graphs, permutation graphs, and trapezoid graphs. Also, we can get a 1.368-approximation algorithm for circular arc graphs [8,11].

Similarly, since the maximum antimatching in a graph corresponds to the maximum clique in the square of its line graph, we can get efficient algorithms for the antimatching problem on chordal graphs [2,7], co-comparability graphs [9,7], and circular arc graphs [8,7].

4. The maximum antimatching problem

In this section, we consider the maximum antimatching problem.

MAXIMUM ANTIMATCHING

Instance: A graph G and a positive integer K.

Question: Does G have an antimatching with at least K edges?

Theorem 2. MAXIMUM ANTIMATCHING *is* **NP**-*complete*.

Proof. The problem is clearly in **NP**. We prove that the following **NP**-complete problem (see [6]) can be reduced to MAXIMUM ANTIMATCHING.

CLIQUE

Instance: A graph H and a positive integer B.

Question: Does H contain a clique of size B?

For every instance (H,B) of CLIQUE we construct an instance (G,K) of MAXIMUM ANTIMATCHING such that G has an antimatching of size K if and only if H has a clique of size B. Let m denote the number of edges of H. Corresponding to every vertex v of H, we add a set A_v of $m+1$ new vertices and connect them to v (see Fig. 3). Let G be the resulting graph and $K = B(m+1)$. Also, let E_v denote the set of edges between v and A_v.

Assume that H has a clique of size B. Clearly, the set of edges of G incident to at least one of the vertices of the clique is an antimatching. This set contains at least $B(m+1) = K$ edges.

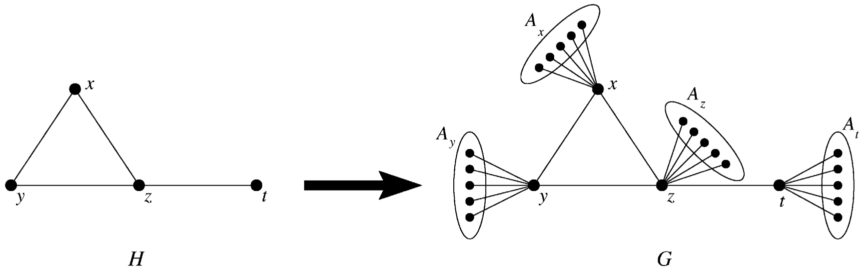

Fig. 3. Reduction from CLIQUE to MAXIMUM ANTIMATCHING.

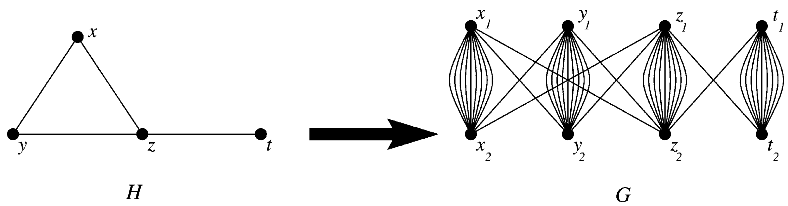

Fig. 4. Reduction from CLIQUE to bipartite MAXIMUM ANTIMATCHING.

Conversely, assume that G has an antimatching of size K. At most m edges of this antimatching are edges of H, and the others are edges from E_v's. Therefore, the antimatching contains at least

$$K - m = B(m+1) - m > (B-1)(m+1)$$

edges from E_v's. Therefore, there are at least B different vertices v such that the antimatching has at least one edge from E_v. This is impossible unless these B vertices form a clique in H. Thus, H has a clique of size B. □

One interesting special case of the antimatching problem is when the input graph is restricted to bipartite graphs. The following theorem shows that this special case is also **NP**-complete if we allow multi-edges.

Theorem 3. MAXIMUM ANTIMATCHING *is* **NP**-*complete for bipartite multigraphs.*

Proof. We use a technique similar to the technique of the proof of Theorem 2 to reduce CLIQUE to this problem. For every instance (H,B) of CLIQUE, we construct a graph G as follows: Corresponding to each vertex v of H, we put two vertices v_1 and v_2 in G and connect them using $2m+1$ edges. For every two adjacent vertices u and v in H, we connect u_1 to v_2 and u_2 to v_1 (see Fig. 4 for an example). Let $K = (2m+1)B$. It is not difficult to see that H has a clique of size B if and only if G has an antimatching of size K. □

5. The antimatching problem for C_4-free graphs

We saw in Section 2 that Strong edge coloring remains **NP**-complete even if we impose the restriction that the graph has an arbitrarily large girth. But the status is different for Maximum antimatching. We will see in this section that this problem can be solved in polynomial time if we restrict the input to be C_4-free. The class of C_4-free graph turns out to have interesting properties with respect to strong edge coloring (see for example [12]). The following theorem characterizes the structure of antimatchings in this class of graphs.

Theorem 4. *Let G be a C_4-free graph and for every vertex v of G, E_v be the set of edges incident to v. Then for every antimatching A in G, at least one of the following propositions hold*:

(a) *There is a triangle uvw such that $A \subseteq E_u \cup E_v \cup E_w$.*
(b) *There is an edge uv such that $A \subseteq E_u \cup E_v$.*
(c) *There is a vertex u such that $A - E_u$ contains at most* 10 *edges.*

It is easy to apply the above theorem to prove the following result.

Theorem 5. *There is a polynomial time algorithm that for every given C_4-free graph G, finds the largest antimatching in G.*

Proof. By Theorem 4 there are only a polynomial number of cases for a maximum antimatching in G. Therefore we can check all possibilities exhaustively as follows.

First, we find a triangle uvw in G for which $\deg(u)+\deg(v)+\deg(w)$ is maximum. It is clear that $A_1 = E_u \cup E_v \cup E_w$ is the largest antimatching for which (a) in Theorem 4 holds. Similarly, we can find the edge uv for which $\deg(u)+\deg(v)$ is maximum. This gives the antimatching $A_2 = E_u \cup E_v$ which is the largest antimatching satisfying (b).

Also, For every pair (u, S), where u is a vertex of G and S is a set of at most 10 edges of $E(G) - E_u$, we check whether S is an antimatching in G. If it is an antimatching, for every edge $e \in E_u$, we check whether the set $S \cup \{e\}$ is an antimatching. Let S' be the set of edges $e \in E_u$ such that $S \cup \{e\}$ is an antimatching. Since every two edges in E_u are of distance at most two, the set $S \cup S'$ is an antimatching in G, and in fact, it is the largest antimatching that contains S and is contained in $E_u \cup S$. Since the number of pairs (u, S) is bounded by a polynomial, it is possible to check every such pair, and find the largest antimatching A_3 for which (c) holds in Theorem 4.

Thus, by Theorem 4, the largest antimatching in G is A_1, A_2, or A_3, whichever is larger; and this antimatching can be found in polynomial time. □

Before proving Theorem 4, we prove the following lemmas.

Lemma 2. *Let G be a C_4-free graph and A be an antimatching in G. If at least three edges of A are incident to some vertex x, then every edge in A has at least one endpoint adjacent to x.*

Proof. Let xy_1, xy_2, xy_3 be three edges of A incident to x. Assume that there is an edge ab in A such that a and b are not adjacent to x. Therefore, since for every $i \in \{1,2,3\}$, xy_i and ab are of distance at most two, y_i must be adjacent to either a or b. Therefore, for some $i_1, i_2 \in \{1,2,3\}$, y_{i_1} and y_{i_2} are both adjacent to the same vertex, say a. Thus, $ay_{i_1}xy_{i_2}$ is a C_4 in G, contradicting our assumption. □

Lemma 3. *Let G be a C_4-free graph and A be an antimatching in G. If x, x', y_1, y_2, y'_1, y'_2 are distinct vertices of G, and the edges xy_1, xy_2, $x'y'_1$, $x'y'_2$ are in A, then x is adjacent to x'.*

Proof. Assume that x is not adjacent to x'. Because of C_4-freeness of G, x cannot be adjacent to both y'_1 and y'_2 at the same time. Therefore, without loss of generality, we can assume that it is not adjacent to y'_1. Similarly, we can assume that x' is not adjacent to y_1. Therefore, since xy_1 and $x'y'_1$ are of distance at most two, y_1 and y'_1 are adjacent. Thus, y_1 cannot be adjacent to y'_2. Therefore, since xy_1 and $x'y'_2$ are of distance at most two, x must be adjacent to y'_2. By a similar argument, x' must be adjacent to y_2. Therefore, $xy'_2x'y_2$ is a C_4 in G, which is a contradiction. □

Lemma 4. *Let G be a C_4-free graph and A be an antimatching in G. Then, A does not contain any matching of size* 8.

Proof. Assume, for contradiction, that $\{u_1v_1, u_2v_2, \ldots, u_8v_8\}$ is a matching of size 8 in A. Since A is an antimatching, there is an edge between one of the endpoints of u_iv_i and one of the endpoints of u_jv_j, for every $i \neq j$. Therefore, without loss of generality, we may assume that u_1 is connected to 4 other endpoints, say u_2, u_3, u_4, u_5. Every vertex in the set $\{u_2, u_3, u_4, u_5\}$ is connected to at most one other vertex in this set, for otherwise these vertices and u_1 will form a C_4. Therefore, we may assume, without loss of generality, that there is no edge between any pair of the vertices $\{u_2, u_3, u_4, u_5\}$, except possibly between u_2u_3 and between u_4u_5. Furthermore, if there is an edge between u_i and v_j for some $i, j \in \{2,3,4,5\}$ $(i \neq j)$, we will get a 4-cycle $u_1u_iv_ju_j$ in the graph, which is impossible. Therefore, for any $i \in \{2,3\}$, $j \in \{4,5\}$, since u_iv_i and u_jv_j are of distance at most two, there must be an edge between v_i and v_j. But this implies that we have the 4-cycle $v_2v_4v_3v_5$ in the graph, which is a contradiction. □

Proof of Theorem 4. Let A be an antimatching in G, and H be the subgraph of G induced by A. Notice that in the following, by "adjacent" we mean adjacent in G, unless otherwise stated. We consider the following two cases separately.

Case 1: *There are at least two vertices whose degrees in H are at least three.* Let u, v be two such vertices. Using Lemma 3 and the fact that u and v cannot have more than one common neighbor, it is easy to see that u and v are adjacent. Now, if $A - E_u - E_v$ is empty, then (b) is true; otherwise, consider an edge ab of $A - E_u - E_v$. By Lemma 2, at least one of the vertices a, b is adjacent to u, and similarly, at least one of them is adjacent to v. If one of them is adjacent to u, and the other is adjacent to v, the vertices u, v, a, b would form a 4-cycle which is impossible. Therefore, either a or b is adjacent to both u and v. Without loss of generality, assume that a is adjacent to both u and v. If there is any other edge $a'b'$ in $A - E_u - E_v$, by a similar argument, one of its endpoints, say a', is adjacent to both u and v. But since G is C_4-free, this is impossible unless $a = a'$. Therefore, $A \subseteq E_u \cup E_v \cup E_a$ and (a) is true.

Case 2: *There is at most one vertex of degree at least three in H*. Let u be a vertex of maximum degree in H. Therefore, the degree of every vertex other than u in H is at most two. This means that $H - E_u$ is a collection of disjoint paths and cycles. By Lemma 3 and C_4-freeness of G, $H - E_u$ cannot contain a cycle and a path of length more than one (i.e., with more than one edge) at the same time. Also, by Lemma 3, $H - E_u$ does not contain more than three paths of length more than one (otherwise the vertices of degree two would form a 4-cycle). Finally, by Lemma 4, $H - E_u$ does not contain any matching with more than 7 edges. These facts imply that $H - E_u$ does not contain more than 10 edges. Therefore, (c) holds in this case. □

Remark 1. The constant 10 in Theorem 4 is not optimal. Using a straightforward (but lengthy) argument, it is possible to replace this constant by 4.

Remark 2. A corollary of the above theorem is that the size of the largest antimatching in a C_4-free graph of maximum degree $\Delta > 1$ is at most $\max\{3\Delta - 3, \Delta + 10\}$. For general graphs, it is an open conjecture that the size of the largest antimatching is at most $\frac{5}{4}\Delta^2$ (see [5]). However, the weaker result that the number of edges of every graph without any induced matching of size 2 is at most $\frac{5}{4}\Delta^2$, was conjectured by Bermond et al. [1] and proved by Chung et al. [3]. This problem has some applications in the design of bus interconnection networks [1].

6. Open problems

In the proof of Theorem 1, the maximum degree of the graph G that is constructed is $K - 1$. One may ask the following question: For every fixed K, what is the minimum value of Δ such that Strong edge coloring is **NP**-complete for graphs of maximum degree Δ? This problem is probably difficult, since the similar problem for vertex coloring is still open.

Determining the computational complexity of Strong edge coloring and Maximum antimatching for planar graphs and partial k-trees is an open question. Notice that (as noted in [9]) it is proved in [10] that the induced matching problem is **NP**-complete for 3-regular planar graphs, so it is natural to expect the strong edge coloring problem to be hard on this class. Also, complexity of Maximum antimatching on simple bipartite graphs is another unsolved problem.

Another natural question is to determine all the values of k for which the result of Theorem 4 (or something similar) holds for C_k-free graphs.

Acknowledgements

The main part of this research was done while I was a graduate student in the Computer Science department of the University of Toronto. I would like to thank my supervisor, Mike Molloy, and Derek Corneil for their helps. Also, I would like to thank the referees for their useful comments.

References

[1] J.C. Bermond, J. Bond, M. Paoli, C. Peyrat, Graphs and interconnection networks: diameter and vulnerability, in: Surveys in Combinatorics (Southampton), London Mathematical Society Lecture Note Series, Vol. 82, ed., E. Keith Lloyd, Cambridge Univ. Press, Cambridge, New York, 1983, pp. 1–30.
[2] K. Cameron, Induced matchings, Discrete Appl. Math. 24 (1989) 97–102.
[3] F.R.K. Chung, A. Gyárfás, Zs. Tuza, W.T. Trotter, The maximum number of edges in $2K_2$-free graphs of bounded degree, Discrete Math. 81 (1990) 129–135.
[4] R.J. Faudree, A. Gyárfás, R.H. Schelp, Zs. Tuza, Induced matchings in bipartite graphs, Discrete Math. 78 (1989) 83–87.
[5] R.J. Faudree, A. Gyárfás, R.H. Schelp, Zs. Tuza, The strong chromatic index of graphs, Ars Combin. 29B (1990) 205–211.
[6] M.R. Garey, D.S. Johnson, Computers and Intractability, A Guide to the Theory of NP-Completeness, W.H. Freeman and Company, San Francisco, 1979.
[7] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York, 1980.
[8] M.C. Golumbic, R.C. Laskar, Irredundancy in circular arc graphs, Discrete Appl. Math. 44 (1993) 79–89.
[9] M.C. Golumbic, M. Lewenstein, New results on induced matchings, Discrete Appl. Math. 101 (2000) 157–165.
[10] C.W. Ko, F.B. Shepherd, Adding an identity to a totally unimodular matrix, Working Paper, London School of Economics Operations Research, LSEOR 94.14, July 1994.
[11] V. Kumar, Approximating Circular arc Coloring and Bandwidth Allocation in all-optical Ring Networks, Lecture Notes in Computer Science, Vol. 1444, Springer, Berlin, 1998, pp. 147–158.
[12] M. Mahdian, The strong chromatic index of C_4-free graphs, Random Structures Algorithms 17 (2000) 357–375.
[13] M. Mahdian, The strong chromatic index of graphs, M.Sc. Thesis, Department of Computer Science, University of Toronto, 2000.
[14] S.T. McCormick, Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem, Math. Programming 26 (1983) 153–171.
[15] A. Sen, M. Huson, A new model for scheduling packet radio networks, Proceedings of IEEE INFOCOM'96, Vol. 3, pp. 1116–1124.